PRESSURE DISTRIBUTION IN NARROW SLOTTED CHANNELS DURING FLOW OF A RAREFIED GAS WITH SLIPPAGE AND A PHASE TRANSITION AT THE WALLS

B. M. Smol'skii, L. Ya. Lyubin,
P. A. Novikov, G. L. Malenko,
and V. I. Svershcheck

UDC 533.6.011.8

Functions for the determination of the pressure in narrow slots are obtained in the case of three-dimensional flow with allowance for the effect of slippage. The effect of slippage on the nature of the pressure distribution in the gap between round disks during the sublimation of ice is demonstrated.

In some industrial vacuum equipment and evaporation–sublimation heat exchangers the process of sublimation and flow of the sublimate vapor is accomplished in narrow slotted channels formed by the sublimating walls.

The modes of flow will vary depending on the height of the slot and the pressure of the vapor in it. Not only can a viscous mode (Kn < 0.01) occur here but also a molecular-viscous mode (Kn > 0.01) and a translational mode (0.1 < Kn < 1). Since the equations of motion and energy of the flow are strictly valid only in the region of small enough Knudsen numbers, and the well-known attempts to transform them for a description of the motion of rarefied gases are very cumbersome [1, 2], the ordinary Navier–Stokes equations were used in the present work with the insertion into the boundary conditions of additional terms taking into account the slippage and temperature jump at the boundary of the stream with the wall surface. Allowance for the effect of slippage made it possible to cover the range of modes of practical interest corresponding not only to small but also to medium Knudsen numbers (from 0.01 to 0.1).

Axially symmetrical flows of vapor during sublimation in a narrow gap between round disks, corresponding to the viscous mode with Kn < 0.01, were studied in [3]. Below we will examine the three-dimensional flows of a rarefied vapor in narrow slotted channels with a phase transition at the walls.

It is assumed that the slotted channel has a flat middle surface and its height 2h satisfies the condition $|\nabla h| \ll 1$, where $\nabla = \bar{i}\partial/\partial x + \bar{j}\partial/\partial y$. The rectangular coordinates 0xy are introduced in the plane of symmetry. The distance from this plane is measured by the coordinate z. Since the height of the slotted channel in this case is small compared with the scale of the vapor flow in the 0xy plane ($h \ll L$), the equations of motion, continuity, and heat conduction can be represented in the following form (separating out the transverse velocity component $\bar{v} = \bar{u} + \bar{k}w$, as in the analysis of flow of an incompressible fluid in a gap [4]):

$$\mu \frac{\partial^2 \bar{u}}{\partial z^2} - \nabla P = \rho(\bar{u}\nabla)\bar{u} + \rho w \frac{\partial \bar{u}}{\partial z} - \mu\nabla^2\bar{u}, \tag{1}$$

$$\frac{\partial P}{\partial z} = \mu \frac{\partial^2 w}{\partial z^2} - \rho w \frac{\partial w}{\partial z} - \rho\bar{u}\nabla w + \mu\nabla^2 w, \tag{2}$$

$$\frac{\partial (\rho w)}{\partial z} + \nabla(\rho\bar{u}) = 0, \tag{3}$$

Institute of Heat and Mass Transfer, Academy of Sciences of the Belorussian SSR, Minsk. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 25, No. 5, pp. 885-892, November, 1973. Original article submitted October 5, 1972.

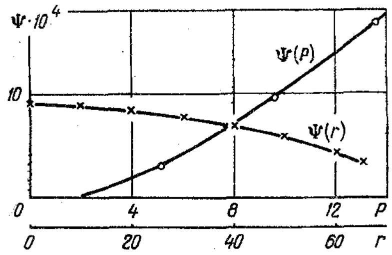

Fig. 1

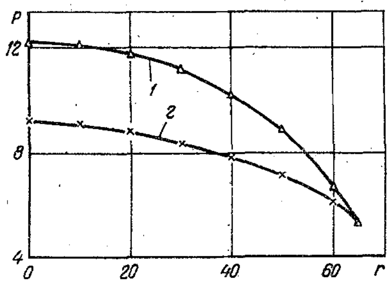

Fig. 2

Fig. 1. Dependence of function Ψ on pressure P, N/m^2 (9)-(12) and on disk radius r, mm (13).

Fig. 2. Variation in pressure P, N/m^2 over length of slot r, mm: 1) from [3]; 2) from Eqs. (9)-(13).

$$\frac{\partial}{\partial z}\left(\lambda\frac{\partial T}{\partial z}\right)=c_p\rho\,\bar{u}\nabla T+c_p\rho w\frac{\partial T}{\partial z}-\nabla(\lambda\nabla T). \tag{4}$$

The conditions corresponding to allowance for slippage and a temperature jump must be satisfied at the channel walls $z = \pm h(x, y)$:

$$u=\frac{2-\sigma}{\sigma}l\left(\frac{\partial\bar{u}}{\partial z}\right)_{z=-h}+\frac{3}{4}\frac{\mu}{\rho T}\nabla T, \tag{5}$$

$$T=T_w+\frac{2-\alpha}{\alpha}\frac{2\gamma}{\gamma+1}\frac{l}{\Pr}\left(\frac{\partial T}{\partial z}\right)_{z=-h} \tag{6}$$

Let us consider the laminar flow of a vapor in a slotted channel corresponding to small Reynolds numbers which characterize these flows and the flows of an incompressible fluid of the Hele–Shaw type. Equations (4) and (6) only are used to estimate the degree of heterogeneity of the temperature field along the height of the channel. For this purpose we introduce the dimensionless coordinates $\xi = x/L$, $\eta = y/L$, and $\zeta = z/h_m$ and the dimensionless velocities $\bar{U} = \bar{v}/U_0$ and $W = w/V_0$, where $h_m = \max h(x, y)$. Neglecting the dependence of the thermophysical parameters on the temperature and using some average values of these parameters for the estimate, we obtain in place of Eq. (4)

$$\frac{\partial^2 T}{\partial\zeta^2}=\mathrm{Pe}\left(\bar{U}\nabla_{\xi\eta}T+W\frac{\partial T}{\partial\zeta}\right)-\frac{h_m^2}{L^2}\nabla_{\xi\eta}^2T, \tag{7}$$

where $\nabla_{\xi\eta} = L\nabla$; the number $\mathrm{Pe} = \mathrm{RePr} = c_p\rho h_m^2 V_0/(\lambda L_1)$ characterizes the ratio of the convective and conductive components of heat exchange in the vapor flowing in the slotted channel if L_1 is the linear scale corresponding to significant relative temperature changes in the 0xy plane. We will assume that $L_1/L = 0(1)$.

If $h_m^2/L^2 \ll \mathrm{Re}$, assuming that $\Pr \lessapprox 1$ for gases, we obtain the following estimate for the ratio of the temperature drops ΔT_h along the height of the slot to the temperature drops ΔT_L along the plane of symmetry: $\Delta T_L : \Delta T_h/\Delta T_L = 0(\mathrm{Pe}) = 0(\mathrm{Re})$. Using the latter result and the condition (6) we can estimate the size of the temperature jump $(T-T_w)/\Delta T_L = 0\ (\mathrm{Kn\,Re})$.

We will use the equation of state for an ideal gas

$$\rho = P/(RT). \tag{8}$$

From (1)-(3) one can obtain an estimate for the ratio of the pressure drops along the height of the slot and in the plane of symmetry: $\Delta P_h/\Delta P_L = 0(h^2/L^2)$. Thus, one can assume that the parameters ρ, T, and P of the vapor are practically constant along the height of the slot and the sublimation at the walls proceeds in a quasi-equilibrium fashion, i.e., the pressure and temperature of the vapor in the slot can be related by the Clapeyron–Clausius equation:

$$T=\frac{\Lambda}{R}F(P),\ F(P)=\frac{RT_*/\Lambda}{1-(RT_*/\Lambda)\ln(P/P_*)}.$$

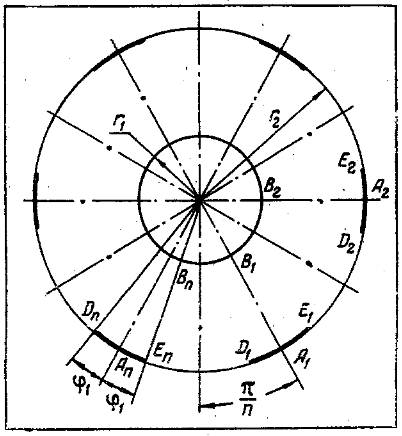

Fig. 3. Diagram of flat slot with obstructions at outer boundary.

Since the ratio of the terms on the right side of Eq. (1) to the terms on the left side has the order of magnitude of Re or h^2/L^2, they can be neglected in the linear approximation to which we are limited here and the following expression can be written for the distribution of the longitudinal velocity component of the vapor over the height of the slot:

$$\bar{u} = -\frac{h^2}{2\mu}\left(1 + 2\frac{2-\sigma}{\sigma}\frac{l}{h} - \frac{z^2}{h^2}\right)\nabla P + \frac{3}{4}\frac{\mu}{\rho T}\nabla T.$$

Since $l = 1.26\mu\sqrt{\gamma}/(\rho a)$ and $a = \sqrt{\gamma RT}$, we obtain

$$\bar{V} = \frac{1}{h}\int_0^h \bar{u}dz = -\frac{h^2}{3\mu}\left[1 + \frac{2-\sigma}{\sigma}\frac{3.78\mu}{\rho h\sqrt{RT}} - \frac{9}{4}\frac{\mu^2}{\rho h^2}\frac{d\ln F(P)}{dP}\right]\nabla P.$$

We can find the pressure distribution in the gap between the subliming surfaces from the equation of material balance

$$\nabla(h\rho V) = J_m,$$

where J_m is the intensity of sublimation, which can be given either as a function of the coordinates $J_m(x, y)$ which is proportional to the specific power of the heating elements, or as a function of the parameters of state of the vapor flowing in the gap [3].

In addition, the pressure P of the vapor must be given in the sections of the contour limiting the region of flow. In the sections corresponding to the construction elements blocking the slotted channel the component of the velocity $\bar{V}$ normal to the contour is equal to zero.

If the height of the slot is constant, the latter equation can be represented in the form

$$\nabla^2\Psi = -\frac{3\mu J_m}{h^3}, \quad \Psi = \int_{P_*}^{P}(\Phi_1 + \Phi_2 + \Phi_3)\,dP, \tag{9}$$

where

$$\Phi_1 = \frac{P}{RT};\ \Phi_2 = \frac{2-\sigma}{\sigma}\frac{3.78\mu}{h\sqrt{RT}};\ \Phi_3 = -\frac{9}{4}\frac{\mu^2}{h^2}\frac{d\ln F(P)}{dP};$$

$$\Psi_1 = \frac{P^2 - P_*^2}{2RT_*} - \frac{1}{\Lambda}\frac{P^2}{2}\ln\frac{P}{P_*} + \frac{P^2 - P_*^2}{4\Lambda}; \tag{10}$$

$$\Psi_3 = \frac{9}{4}\frac{\mu^2}{h^2}\ln\left(1 - \frac{RT_*}{\Lambda}\ln\frac{P}{P_*}\right). \tag{11}$$

If $|(RT_*/\Lambda)\ln(P/P^*)| \ll 1$, then in the expansion

$$\sqrt{1 - \frac{RT_*}{\Lambda}\ln\frac{P}{P_*}} = 1 - \frac{1}{2}\frac{RT_*}{\Lambda}\ln\frac{P}{P_*} + \cdots$$

one can be confined to the first few terms. Then

$$\Psi_2 = \frac{2-\sigma}{\sigma}\frac{3.78\mu}{h\sqrt{RT_*}}\left[P - P_* - \frac{1}{2}\frac{RT_*}{\Lambda}P\left(\ln\frac{P}{P_*} + \frac{P_*}{P} - 1\right)\right.$$
$$\left. - \frac{1}{8}\left(\frac{RT_*}{\Lambda}\right)^2 P\left(\ln^2\frac{P}{P_*} - 2\ln\frac{P}{P_*} - 2\frac{P_*}{P}\right) - \cdots\right]. \tag{12}$$

Thus, if $J_m = J_m(x, y)$, then in the present case the determination of the pressures in the slot comes down to a search for the solution Ψ of the boundary problem for the Poisson Equation (9) in the region bounded by the contour Γ, which consists of the sections Γ_j' and Γ_k'' (j, k = 1, 2, . . .). In the sections Γ_j' the values of the function corresponding to the given pressures P_j are known. In the sections Γ_k'' the derivative in the direction normal to the contour is $\partial\Psi/\partial n = 0$.

As an example let us examine the flow of a vapor during sublimation in an open gap between round disks of radius r_2. The external pressure is assumed to equal P', i.e., $\Psi = \Psi' = \Psi(P')$ when $r = r_2$. The pressure distribution of the vapor when J_m = const corresponds to the following function Ψ:

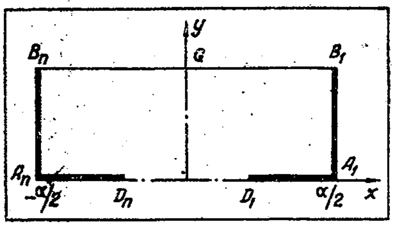

Fig. 4

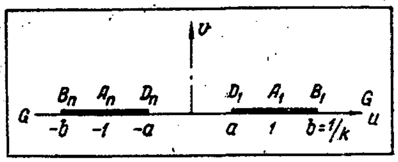

Fig. 5

Fig. 4. Conformal mapping of region $A_1B_1B_nA_n$ onto rectangle.

Fig. 5. Conformal mapping of rectangle (Fig. 4) onto upper half-plane.

$$\Psi = \Psi' + \frac{3\mu J_m}{4h^3}(r_2^2 - r^2). \tag{13}$$

Experimental data obtained in [3] in a study of the flow of a vapor-sublimate in the gap between round disks were used to estimate the effect of slippage at the walls of the channel. The initial data for the calculation are: height of slot $2h = 2$ mm, diameter of disks $2r_2 = 130$ mm, intensity of sublimation $J_m = 0.49 \cdot 10^{-4}$ kg/m$^2\cdot$sec, pressure in chamber $P = 5.33$ N/m^2, fraction σ of molecules reflected diffusely from walls taken as 0.9. In this mode of flow $Kn \approx 0.1$.

The order of the calculations was as follows: 1) we choose a P_* and a T_* corresponding to it on the saturation curve for water vapors over ice; 2) we set an arbitrary temperature T and determine the pressure P from the Clapeyron–Clausius equation with a calculation which encompasses the assumed range of variation in pressure over the length of the slot; 3) using Eqs. (9)-(13) we construct graphs of $\Psi = \Psi(P)$ and $\Psi = \Psi(r)$ (Fig. 1); 4) from the graphs of $\Psi = \Psi(P)$ and $\Psi = \Psi(r)$ we find $P = P(r)$ (Fig. 2).

The data obtained in [3] without allowing for slippage at the walls of the slot are also presented in Fig. 2 for comparison (curve 1). It is seen from the figure that ignoring the slippage effect in our case leads to a pressure about 1.3 times higher at the center of the disks.

In the present work we neglect the resistance of the phase transition ΔP_{ph} because, as a calculation shows, the hydraulic resistance ΔP_h is much greater in the range of modes of flow under consideration. For example, for $r_2 = 65$ mm, $2h = 2$ mm, $\beta \approx 1$, and $\sigma \approx 1$:

$$\text{for } \mathrm{Kn} = 0.1 \quad \frac{\Delta P_{ph}}{\Delta P_h} \approx 0.008,$$

$$\text{for } \mathrm{Kn} = 0.01 \quad \frac{\Delta P_{ph}}{\Delta P_h} \approx 0.05.$$

In some sublimation heat-exchange equipment of the cassette type [5] the construction elements may be located directly at the end face of the channels, blocking the space for the escape of vapor and exerting a certain effect on the pressure distribution along the length of the slot. Therefore as an example let us consider the sublimation in the gap between round disks (Fig. 3) closed along the arcs D_kE_k ($k = 1, 2, \ldots, n$) distributed uniformly along the circumference $r = r_2$, with a radial distribution of intensities ($J_m = J_m(r)$). It is assumed that at the inner edge ($r = r_1$) the pressure is $P = P_1(\Psi(P_1) = \Psi')$, and at the open sections of the outer boundary ($r = r_2$) the pressure is $P = P_2(\Psi(P_2) = \Psi'')$.

The function Ψ is sought in the form of the sum $\Psi_0 + \Psi_r$, where

$$\Psi_0 = \Psi' + \frac{3\mu}{h^3}\int_{r_1}^{r}\left[\int_{r_*}^{r_2} J_m(r_0)\, r_0\, dr_0\right]\frac{dr_*}{r_*}, \tag{14}$$

while the harmonic function Ψ_r because of the symmetry of flow is determined in the region $A_1B_1B_nA_n$, at the boundary of which it satisfies the conditions

$$\begin{gathered}\Psi_r = 0 \text{ at } B_nB_1;\ \Psi_r = \Psi'' - \Psi_0(r_2) \text{ at } D_nD_1,\\ \partial\Psi_r/\partial r = 0 \text{ at } D_1A_1 \text{ and } D_nA_n;\\ \partial\Psi_r/\partial\varphi = 0 \text{ at } A_1B_1 \text{ and } A_nB_n.\end{gathered} \tag{15}$$

The function $z = -i \ln(R/r_2) = \varphi + i \ln(r_2/r)$ conformally maps this region onto a rectangle (Fig. 4 ; $Re = re^{i\varphi}$) which is conformally mapped onto an upper half-plane (Fig. 5) by the function

$$z = c\int\limits_0^w \frac{dw}{\sqrt{(1-w^2)(1-k^2w^2)}}.$$

The coefficients k and c are determined from the equations (Figs. 4 and 5)

$$\alpha = 2c\int\limits_0^1 \frac{du}{\sqrt{(1-u^2)(1-k^2u^2)}} = 2cK(k) = 2\pi/n,$$

$$\beta = c\int\limits_1^{1/k} \frac{du}{\sqrt{(u^2-1)(1-k^2u^2)}} = cK(k') = \ln\frac{r_2}{r_1},$$

where $K(k)$ and $K(k')$ are complete elliptical integrals; $k' = \sqrt{1-k^2}$. Consequently, $c = \pi/[nK(k)]$. The parameter k can be found, knowing $\varkappa = K(k')/K(k) = (n/\pi)\ln(r_2/r_1)$, from a table of $k^2 = f(q)$ [6], where $q = e^{-\pi\varkappa} = (r_1/r_2)^n$.

The search for Ψ_r comes down to the solution of the Keldysh–Sedov problem through the introduction of the analytical functions $f = \Psi_r + i\Psi_i$ and $f_1 = df/dw = \varphi + i\Psi$ [7]. Since $\varphi = \partial\Psi_r/\partial u$, $\Psi = -\partial\Psi_r/\partial v$, in place of (15) one can write $\varphi = 0$ on GB_n, D_nD_1, and B_1G and $\Psi = 0$ on B_nD_n and D_1B_1.

Thus,

$$f_1 = \frac{\gamma_0 + \gamma_1 w}{\sqrt{(w^2-a^2)(w^2-b^2)}}, \quad a = u_{D_1} = \mathrm{sn}\left[\left(1 - \frac{n\varphi_1}{\pi}\right)K(k),\ k\right],$$

$$b = 1/k, \tag{16}$$

$$\Psi_r = \mathrm{Real}\, f = \mathrm{Real}\int\limits_0^w f_1(w)\,dw + \Psi' - \Psi_0(r_2).$$

Because of the symmetry of Ψ_r relative to the 0v axis $\gamma_0 = 0$. The coefficient γ_1 is determined from the condition

$$\Psi_r(B_1) - \Psi_r(D_1) = \Psi_0(r_2) - \Psi'' = \gamma_1\int\limits_a^b \frac{u\,du}{\sqrt{(u^2-a^2)(u^2-b^2)}} = \frac{\pi\gamma_1 i}{2}, \text{ i.e., } \gamma_1 = \frac{2i}{\pi}[\Psi'' - \Psi_0(r)].$$

Consequently,

$$f = [\Psi'' - \Psi_0(r_2)]\left(1 + \frac{2i}{\pi}\ln\frac{\sqrt{a^2-w^2} - \sqrt{b^2-w^2}}{a-b}\right).$$

When $P_1 \geq P_2$ the points corresponding to the maximum pressure of the vapor will lie at the openings A_kB_k ($a < 1 < u < b$). Along such an opening the function Ψ is determined by the equation

$$\Psi = \Psi_0[r(u)] + [\Psi'' - \Psi_0(r_2)]\left(1 - \frac{2}{\pi}\,\mathrm{arctg}\sqrt{\frac{u^2-a^2}{b^2-u^2}}\right).$$

The radius corresponding to the point (u, 0) is equal to

$$r = r_2 \exp\left[-\frac{\pi}{nK(k)}F\left(\frac{\sqrt{u^2-1}}{k'u},\ k'\right)\right].$$

Here $F(w, k)$ is an elliptical integral of the first kind.

The flow rate of the vapor through the open part of the contour $r = r_2$ is determined by the expression

$$Q = \frac{2nh^3}{3\mu}\int\limits_0^a \frac{\partial\Psi}{\partial v}du, \quad \left.\frac{\partial\Psi}{\partial v}\right|_{F_nF_1} = \frac{\partial\Psi_r}{\partial v} = -\Psi = \mathrm{Im}\, f_1 = \frac{2}{\pi}[\Psi'' - \Psi_0(r_2)]\frac{u}{\sqrt{(a^2-u^2)(b^2-u^2)}}.$$

Thus,

$$Q = \frac{2nh^3}{3\pi\mu}[\Psi'' - \Psi_0(r_2)]\ln\frac{1+ak}{1-ak}.$$

NOTATION

J_m	is the sublimation intensity;
$2h$ and $2r_2$	are the height and length of slot;
T, P, ρ, and μ	are the temperature, pressure, density, and coefficient of dynamic viscosity of vapor in slot;
T_* and P_*	are the temperature and pressure of vapor on saturation line;
Λ	is the latent heat of sublimation;
R	is the gas constant;
λ and α	are the coefficients of thermal conduction and accommodation;
σ	is the fraction of molecules diffusely reflected from wall;
l	is the free path length of molecules;
$\gamma = c_p/c_v$;	
v	is the transverse component of velocity of vapor perpendicular to middle surface;
u	is the longitudinal component of velocity of vapors;
V	is the velocity of vapor;
$Kn = l/2h$	is the Knudsen number;
$Re = \rho V h^2/\mu L$	is the Reynolds number;
a	is the speed of sound;
L	is the characteristic scale in plane of slot;
β	is the condensation coefficient.

LITERATURE CITED

1. D. Burnett, Proc. London Math. Soc., 40, 382 (1935).
2. S. Chapman and T. G. Cowling, The Mathematical Theory of Nonuniform Gases, Cambridge Univ. Press, London (1939).
3. P. A. Novikov, G. L. Malenko, L. Ya. Lyubin, and V. I. Balakhonova, Inzh.-Fiz. Zh., 22, No. 5 (1972).
4. A. S. Povitskii and L. Ya. Lyubin, Hydrodynamics and Heat and Mass Transfer in Weightlessness [in Russian], Moscow (1972).
5. W. C. Kincaide, Mech. Engin., 87, No. 1 (1965).
6. E. Jahnke and F. Émde, Tables of Functions with Equations and Curves [Russian translation], Gostekhizdat (1948).
7. M. A. Lavrent'ev and B. V. Shabat, Methods of the Theory of Functions of a Complex Variable [in Russian], Fizmatgiz, Moscow (1958).